The A2 automaton

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Abstract

Cellular automata are a computational model first described by von Neuman [3]. One of the most remarkable example of a CA is the "Game of Life" (GOL) defined by Conway in 1977. A slightly modified version of the "Game of Life" is represented and explored in this paper.

1 Introduction

This paper describes a cellular automaton, called *A2*. Within similiar kinds of automata, this is one of the simplest automaton possible. In this paper the author tries to establish a framework for defining the automata, deliver the result of observations made and provide (yet unproven) theorems.

2 A framework for boolean matrix based cellular automaton

This chaper gives some basic definitions of the terms and denotions used later in this paper. Most of the definitions and terms of this chapter can be applied to most two-dimensional cellular automaton (CA). Those definitions are made with respect to existing CA definitions [4], although with minor differences.

Definition 1 (Population)**.** Let ϕ_n be the set of all $n \times n$ matrices over $\{0, 1\}$

$$\phi_n : \{x \quad | \quad x = (x_{ij}) \quad n \times n \text{ matrix}, x_{ij} \in \{0, 1\}\}$$

A matrix $P \in \phi_n$ is named *population of size n*. Further a $c_{ij} \in P \in \phi_n$ is a *cell of P*; the cell is called *alive* if $c_{ij} = 1$, *dead* otherwise. Let the zero matrix $0 \in \phi_n$ denote an empty population.

In order to compute the next generation of a cell, some criteria for the next generation state has to be established. Intuitively the state (either alive or dead) of a cell is determined by its environment. E.g. if there are too many alive cells surrounding, there might be too less "food" available, so the cell will die in the next generation. Put to the world of cellular automata, this *environment* refered to above will be defined here to be the Moore neighborhood [5] with range $r = 1$.

Definition 2 (Moore neighborhood)**.** Let be $P \in \phi_n$ and $P = (c_{ij})$. The *Moore neighborhood* of a cell c_{ij} consists of all cells directly surounding c_{ij}:

$$\begin{aligned} U_p(i, j) \quad = \quad & \{(k, l) \quad | \quad 1 \leq k \leq n, 1 \leq l \leq n, \\ & k \in \{i - 1, i, i + 1\}, \\ & l \in \{j - 1, j, j + 1\}, \\ & (k, l) \neq (i, j)\} \end{aligned}$$

Graphically the Moore neighborhood can be shown as follows (where the white field is c_{ij} and the gray fields are composing the Moore neighborhood):

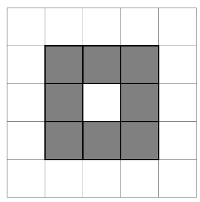

Figure 1: illustration of the Moore neighborhood

2.1 Reproduction

The pure existence of a population and it's cells can be considered trivial. It's the dynamic component inherent in CA (or automata in general). Within

CA this dynamic is produced by the reproductional act which is one of the most significant features of life in general [1]. Implementing this behaviour in a discrete world [2], we define the reproduction as a sequence of populations.

Definition 3. Are $P_0, P_1, P_2, \ldots, P_k$ a sequence of populations of size n: $< |P_k| >$, so P_0 is called *initial population.* Let $\lambda_k = \{P_i \quad | \quad i < k\}$ denote the ancestors of P_k where P_k is the $k-th$ generation of the sequence.

We might also define the sequence of populations recursively. With P_0 given:

$$\begin{aligned} P_{k+1} &= f(P_k) \\ f : \phi_n &\longrightarrow \phi_n \end{aligned}$$

where f is called *reproductional mapping* (RM) (also called *Local function*, [4]). That reproductional mapping is a specific attribute of an individual CA.

2.2 Defining concrete CA

As the previous chapters intuitively make clear, there are three parameters which distinguish such CA within the given framework.

- population size n
- initial population P_0
- reproductional mapping f

So we eventually define a CA as a tripple of those three attributes (at least within this paper).

Definition 4 (Cellular automaton)**.** Is $n \in N$, P_0 a population of size n and f a reproductional mapping, then we denote

$$C = (n, P_0, f)$$

to be a *cellular automaton.*

3 From GOL to A2

All definitions given in chapter 2 would also be suitable for describing Conways "Game of life". In order to do this, yet another notation has to be introduced. So let the following denote the amount of alive cells within a Moore neighborhood

$$\begin{aligned} \alpha : \mathbb{N}^2 &\longrightarrow \mathbb{N} \\ (i,j) &\longmapsto \sum_{(k,l) \in U(i,j)} x_{kl} \end{aligned}$$

Further $\zeta : \mathbb{N}^2 \longrightarrow \mathbb{N}, (i,j) \longmapsto W$ with $W = \{\perp, \top\}$ denotes whether a cell c_{ij} is alive or not.

Using the denotions above the RM of the "Game of Life" can be written like this

$$\begin{aligned} f : \phi_n &\longrightarrow \phi_n \\ x = (x_{ij}) &\longmapsto f(x) = (y_{ij}) \\ \text{where} \quad y_{ij} = & \begin{cases} 1 & \text{when } \varphi(i,j) \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

where $\varphi(i,j) = \big([a(i,j) = 2 \vee a(i,j) = 3] \wedge \zeta(i,j)\big) \vee \big(a(i,j) = 2 \wedge \zeta(i,j)\big)$. After having this established the question arises: what happened if we reduced φ to a bare minimum expression like $\varphi(i,j) : a(i,j) = 2$ (hence the name: A2). So we can finally define the reproductional mapping of the A2 automaton as follows:

Definition 5 (RM of A2)**.**

$$\begin{aligned} f : \phi_n &\longrightarrow \phi_n \\ x = (x_{ij}) &\longmapsto f(x) = (y_{ij}) \\ \text{where} \quad y_{ij} = & \begin{cases} 1 & \text{when } \sum\limits_{(k,l) \in U(i,j)} x_{kl} = 2 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Notice that we could also define y_{ij} as follows:

$$y_{ij} \quad = \quad \begin{cases} 1 & \text{when } \sum\limits_{k=i-1}^{i+1} \sum\limits_{l=j-1}^{j+1} x_{kl} - x_{ij} = 2 \\ 0 & \text{otherwise} \end{cases}$$

Eventually the complete A2 automaton consists of the tripple described above $A2 = (n, P_0, f)$ with f being defined in 5 and n as well as P_0 being arbitrary but firmly.

4 Investigation of the A2 automaton

With such a specific requirement for life as it exists within the A2 automaton, one could expect that virtualy any initial population P_0 would become the empty population after a small amount of generations. In fact this is at least not the case for the following initial population

Definition 6.

$$\begin{aligned} P_0 &\in \phi_n \\ P_0 &= (c_{ij}) \\ \text{where} \quad c_{ij} &= \begin{cases} j \mod 2 & \text{when } j = 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

hence P_0 being

$$P_0 = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \end{pmatrix}$$

Therefrom the question arises, does this population ever become the empty population. So let us describe the following theorem

Theorem 1. Let be $A2 = (n, P_0, f)$ with $n \geq 5$, P_0 as defined in 6 and f as defined in 5. For the sequence of generations is

$$0 \notin \lambda_{2^{(n^2)}-1}$$

4.1 Ideas for proof, informal

With the A2 automaton originating in the GOL, one could expect to find oscilating patterns as they tend to occur in the game of life. In order to identify such patterns and to be able to do computational experiments with the A2 automaton one has to be able uniquely identify a population. This can be achieved by interpreting a population as stream of bits (with all row vectors appended to each other), resulting in a binary number. So the population

$$P = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

becomes $(0101000101000001)_2$ or 20801 in decimal. While searching for reappearing patterns 500,000 generations of the A2 automaton were computed, their "fingerprint" was taken and plotted in a phase plot. Clearly some reoccuring pattern emerges, altough it seems that this behaviour can not be exploited for proof it may rise the attention of the reader for this automaton.

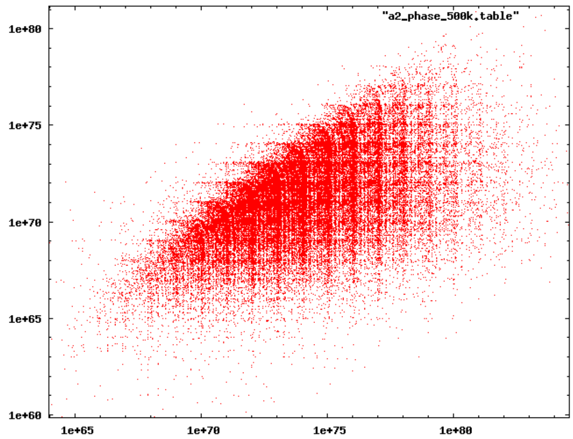

Figure 2: Phase plot of 500,000 A2 generations

5 Conclusion

As this paper has shown, within the area of cellular automata interessting behavior can be observed. Especially the A2 automaton, mostly due to its simplicity is worth further exploration. Also the theorem above is still unproven and therefore leaves room for further research.

References

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[5] E. W. Weisstein, *Moore neighborhood*, MathWorld – A Wolfram Web Resource. http://mathworld.wolfram.com/MooreNeighborhood.html, 2009.